UDEM-GPP-TH-96-35

Solitons in a Baby-Skyrme model with invariance under area preserving diffeomorphisms

T. Gisiger and M. B. Paranjape

Groupe de physique des particules, Laboratoire de physique nucléaire, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, Québec, Canada, H3C 3J7

Abstract

We study the properties of soliton solutions in an analog of the Skyrme model in 2+1 dimensions whose Lagrangian contains the Skyrme term and the mass term, but no usual kinetic term. The model admits a symmetry under area preserving diffeomorphisms. We solve the dynamical equations of motion analytically for the case of spinning isolated baryon type solitons. We take fully into account the induced deformation of the spinning Skyrmions and the consequent modification of its moment of inertia to give an analytical example of related numerical behaviour found by Piette *et al.*[1]. We solve the equations of motion also for the case of an infinite, open string, and a closed annular string. In each case, the solitons are of finite extent, so called "compactons", being exactly the vacuum outside a compact region. We end with indications on the scattering of baby-Skyrmions, as well as some considerations as the properties of solitons on a curved space.

1. Introduction

The baby-Skyrme model is a useful laboratory for studying soliton physics[1−3]. It is the 2+1 dimensional analog of the model which describes the low energy chiral dynamics of

QCD[5], the usual Skyrme model[4]. The model has direct applications in condensed matter physics[6] where baby-Skyrmions give an effective description in quantum Hall systems. In such systems, the dynamics are governed by the spin stiffness term, the Coulomb interaction and the Zeeman interaction. In the relativistic analog of this system the baby Skyrme model is governed by the Lagrangian[1]

$$L = \frac{1}{2}\int d^2x \left[f_\pi \, \partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{2}\Big(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}\Big)^2 - \mu^2(\hat{n}-\vec{\phi})^2\right] \tag{1}$$

where $\vec{\phi}$ is a unit scalar iso-vector field, $\vec{\phi}\cdot\vec{\phi}=1$. This theory admits stable topological soliton solutions. The kinetic energy corresponds to the spin stiffnesss term and the mass term corresponds to the Zeeman interaction, the correspondence being exact for the static sector. The Skyrme term (with four derivatives) is analogous to the Coulomb term (also with four derivatives) and both serve to stabalize from collapse topological configurations which yield Skyrmion solutions. In this paper we will consider the model arising in the limiting case when $f_\pi \to 0$. This corresponds equivalently to $\mu \to +\infty$ with an appropriate rescaling. In the quantum Hall system, this corresponds to the limit $B \to +\infty$ (at finite Landé g factor):

$$L = -\frac{1}{2}\int d^2x \left[\frac{1}{2}\Big(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}\Big)^2 + \mu^2(\hat{n}-\vec{\phi})^2\right]. \tag{2}$$

We have already introduced this model in our previous letter[7], it is our aim in this paper to further elaborate on its properties.

This type of model has been studied by Tchrakian *et al* [8] for a complex scalar field, with a somewhat modified potential. Here, for the static sector they have essentially proven that the model is integrable. In their model there exists a Bogomolnyi bound which is saturated and renders the differential equation to be solved first order. As we shall see below there are certain similarities between the model treated here and their model, however, the corresponding Bogomolnyi bound here is not saturated. Hence it is not evident that our model is also integrable for the static sector.

This Lagrangian admits the following symmetries. It is clearly Poincaré invariant

(Lorentz and translational invariance). In addition, because the mass term picks the vacuum direction $\hat{n}$, the $O(3)$ invariance is explicitly broken to $O(2)$, which corresponds to isorotations about $\hat{n}$:

$$\vec{\phi} \rightarrow R_3 \cdot \vec{\phi} \qquad\qquad R_3 \cdot \hat{n} = \hat{n} \tag{3}$$

A more interesting symmetry comes from invariance under area preserving diffeomorphisms for the static energy

$$E = \frac{1}{2}\int d^2x \left[\frac{1}{4}\left(\epsilon_{ij}\partial_i\vec{\phi} \times \partial_j\vec{\phi}\right)^2 + \mu^2(\hat{n} - \vec{\phi})^2\right] \tag{4}$$

where we see that the vacuum configuration $\vec{\phi} = \hat{n}$ has zero energy. Now if we change the coordinates $x^i \rightarrow x'^i$ with an area preserving diffeomorphism

$$det\left(\frac{\partial x'^i}{\partial x^j}\right) = 1 \tag{5}$$

then the integration measure does not change and

$$\epsilon_{ij}\partial_i\vec{\phi} \times \partial_j\vec{\phi} \rightarrow det\left(\frac{\partial x'}{\partial x}\right)\epsilon_{ij}\partial_i\vec{\phi} \times \partial_j\vec{\phi} \tag{6}$$

which also does not change since the determinant is 1. This symmetry is infinite dimensional, therefore there is an infinite dimensional degeneracy in the energies of the solutions. Intuitively each solution can be deformed like an incompressible fluid to any shape imaginable. This should also translate into an infinite degeneracy of the ground state in the quantum theory, which is a persistent feature of microscopic models describing quantum Hall systems.

We end this section by writing the static energy as follows:

$$E = \frac{1}{2}\int d^2x \left[\left(\frac{1}{2}\,\epsilon_{ij}\,\partial_i\vec{\phi} \times \partial_j\vec{\phi} \pm \mu(\hat{n} - \vec{\phi})\right)^2 \pm 2\mu B(\vec{x})\right] \tag{7}$$

where the extra cross term is a total divergence and vanishes upon integration and:

$$B = \frac{1}{4\pi}\int d^2\vec{x}\; B(\vec{x}) = \frac{1}{8\pi}\int d^2\vec{x}\; \epsilon_{ij}\vec{\phi}\partial_i\vec{\phi} \times \partial_j\vec{\phi}. \tag{8}$$

This expression shows that the energy of a solution is always larger or equal to $4\pi\mu|B|$, giving the Bogomolnyi type bound of our model:

$$E = 4\pi\mu|B| \qquad \text{if and only if} \qquad \frac{1}{2}\,\epsilon_{ij}\,\partial_i\vec{\phi}\times\partial_j\vec{\phi}\pm\mu(\hat{n}-\vec{\phi}) = 0. \tag{9}$$

The plan of this paper is as follows. We start with an analysis of the Skyrmion type solution, including its quantum rotational spectrum taking into account the deformation due to the centripetal acceleration. To our knowledge the back reaction of the rotation on the form of the soliton has never been taken into account in obtaining the quantum rotational spectrum. Such an analysis would be most interesting for the Skyrme model of nucleons. We can do so because we can analytically solve the full equations of a steadily rotating baby-Skyrmion. Afterwards, we continue with our string solutions, already exhibited in our previous letter[7] and present closed string solutions which are physically more realistic. We then present some indications on the low energy scattering of baby-Skyrmions. We end with a section where we consider our model on curved space.

2. The baby-Skyrmion

The configuration space of the model is comprised of maps from the plane R^2 to the target space S^2. Taking coordinates Θ, Φ on the target sphere (corresponding to the usual spherical polar coordinates) the best known solution is the rotationally symmetric solution given by:

$$\begin{aligned}\Theta &= f(r)\\ \Phi &= N\theta\end{aligned} \tag{10}$$

where r, θ are the usual polar coordinates on the plane. This yields the ansatz[9]

$$\vec{\phi} = (\sin f(r)\cos N\theta, \sin f(r)\sin N\theta, \cos f(r)) \tag{11}$$

where N is an integer, and is equal to the baryon number of the configuration. $f(r)$ has to be π at the origin and 0 at infinity to obtain $B = N$.

Replacing (11) into the energy yields the equation of motion:

$$N^2 f''(r)\frac{\sin^2 f(r)}{r} + N^2 f'(r)^2\frac{\sin 2f(r)}{2r} - N^2 f'(r)\frac{\sin^2 f(r)}{r^2} - \mu^2 r\sin f(r) = 0 \tag{12}$$

This complicated non-linear equation, quite surprisingly, can be integrated analytically. By factoring $\sin f(r)$ we see that the first 3 terms of the equation are a total derivative. This yields the equations (integrated)

$$\sin f(r) = 0 \tag{13}$$

$$1 - \cos f(r) = \frac{\mu^2 r^4}{8N^2} + \frac{ar^2}{2} + b \tag{14}$$

where a and b are integration constants.

Finiteness of energy and integer baryon number B implies that $f(r)$ has to be 0 far from the origin. Usually this is achieved via an exponential or some inverse power of r. In our case, this limit has to be attained in an unexpected fashion. For some $r = r_0$, f becomes exactly zero. This is possible if $a \leq -\sqrt{2b}\,\mu|N|$ (reproducing in (14) the polynomial generally known as representing a mexican hat). $f(0) = \pi$ imposes that $b = 2$ hence the critical value of a is $-2\mu|N|$. The function $f(r)$ is defined by (14) only for $r \leq r_0$. For $r \geq r_0$ the function obeys equation (13): f is zero. This effectively "patches" the vacuum to the exterior of the soliton. To determine the values of a and consequently r_0, we minimize the energy. We note that since the soliton is of finite size, and neither $f(r)$ nor $f'(r)$ diverge anywhere, the energy of the baby-Skyrmion is finite. Physical intuition, verified by explicit computation, tells us that since the energy density is a function of $(1 - \cos f)$ and its first derivative, the total energy is minimum if $(1 - \cos f)$ attains zero at r_0 with a zero derivative. This gives the values $a = -2\mu|N|$ and $r_0 = 2\sqrt{|N|/\mu}$ and defines $f(r)$ by

$$f(r) : \begin{cases} 1 - \cos f(r) = \frac{\mu^2 r^4}{8N^2} - \frac{\mu r^2}{|N|} + 2 & \text{if} \quad r \leq 2\sqrt{\frac{|N|}{\mu}} \\ 0 & \text{if} \quad r > 2\sqrt{\frac{|N|}{\mu}} \end{cases} \tag{15}$$

The energy of the Skyrmion can be calculated analytically:

$$E = \frac{4}{3}\, 4\pi\mu|N|. \tag{16}$$

This does not saturate the Bogomolnyi bound of the model. The situation is analogous to the Skyrme model in 3+1 dimensions where it is also possible to find a Bogomolnyi bound,

which is not staurated by the minimum energy configuration. There, however, it is possible to define the model on S^3, which allows one to take advantage of the natural geometrical interpretation of the energy functional to saturate the bound by fixing the size of the S^3. No such interpretation is possible here, as the mass term destroys the symmetry of S^2 hence ruining the geometric interpretation. We show this in detail in the last section.

The fact that the energy is proportional to the baryon number of the solution implies that the energy of 2 sufficiently separated baby-Skyrmions of baryon number B_1 and B_2 is equal to the energy of a localized lump of baryon number $|B_1| + |B_2|$. This gives support to the possibility that there exists a sequence of deformations transforming the two configurations into one another for the case $(B_1, B_2 > 0)$. This would make possible an analytical analysis of low energy baby-Skyrmion scattering. We will say more about scattering later in this paper.

We end this section on static baby-Skyrmions by computing their area. We find trivially:

$$A = \frac{4\pi|N|}{\mu} \tag{17}$$

which is also proportional to the baryon number. So the area of a $|B_1|+|B_2|$ baby-Skyrmion is equal to the sum of the individual areas of the $|B_1|$ and $|B_2|$ baby-Skyrmions. This, together with the energy considerations given above, also points towards a baby-Skyrmion baby-Skyrmion scattering possibly parametrized by a sequence of area preserving diffeomorphisms.

3. Spinning baby-Skyrmion

We now consider the problem of a baby-Skyrmion spinning (about the z direction) and the computation of the corresponding (semi-classical) quantum spectrum. The usual assumption in this type of calculation is that the soliton is sufficiently "stiff" as not to deform enough to significantly modify the energy spectrum. This "rigid rotor" approximation has been used frequently but seldom checked, since this requires the ability to compute the instantaneous soliton profile for any angular velocity (which is no easy task even in a purely numerical framework). This problem was first analysed by Piette *et al.*[1] for the full model (including

kinetic term). They computed numerically the deformation and energies of spinning baby-Skyrmions. We will see that our model possesses analytical solutions for spinning baby-Skyrmions and this enables us to obtain the semi-classical approximation to the quantum rotation energy spectrum of the baby-Skyrmion taking fully into account the deformation necessary to produce the force which maintains the centrifugal acceleration.

The obvious ansatz for a rotating baby-Skyrmion is given by

$$\vec{\phi} = (\sin f(r) \cos (N\theta - \omega t), \sin f(r) \sin (N\theta - \omega t), \cos f(r)). \tag{18}$$

This ansatz has already been used by Wilczek and Zee[9] to explore the connection between fractional spin and exotic statistics, and by Piette *et al*[1].

We will proceed in the following manner. We first extract the instantaneous profile of the soliton as a function of ω from the expression of the energy which then enables us to compute (in principle) the (instantaneous) moment of inertia and total energy of the soliton. The quantum energy spectrum can be obtained semi-classically by using the Bohr-Sommerfeld quantization condition which picks out the allowed values of the angular velocity ω. The ω dependence of the moment of inertia modifies the allowed ω values (and the corresponding energy levels) from the ones obtained with the rigid body approximation.

Replacing the ansatz (18) into the Lagrangian gives

$$\begin{aligned} L = \pi \textstyle\int_0^{+\infty} & dr r \sin^2 f(r) f'(r)^2 \omega^2 \\ & -\pi \textstyle\int_0^{+\infty} dr \left[\frac{\sin^2 f(r)}{r} f'(r)^2 N^2 + 2\mu^2 (1 - \cos f(r)) r \right] \end{aligned} \tag{19}$$

after integrating over the angle θ. It appears as if the only trace of the ongoing rotation of the particle is the paramater ω in the kinetic part of the energy, but there is of course the implicit ω dependence in $f(r)$. It is through that ω dependence of $f(r)$ that deformations enter the problem.

We vary $f(r)$ and obtain the following equation of motion:

$$\begin{aligned} \sin f(r) \cos f(r) f'(r)^2 \Big(\tfrac{N^2}{r} - \omega^2 r\Big) + \sin^2 f(r) f''(r) \Big(\tfrac{N^2}{r} - \omega^2 r\Big) - \\ \sin^2 f(r) f'(r) \Big(\omega^2 + \tfrac{N^2}{r^2}\Big) - \mu^2 r \sin f(r) = 0 \end{aligned} \tag{20}$$

As in the case of the static baby-Skyrmion, we can analytically find the solutions of this equation:

$$f(r):\begin{cases} 1-\cos f(r)=\frac{\mu^2}{2}\Big[-\frac{r^2}{2\omega^2}+\frac{(r_\omega{}^2\omega^2-N^2)}{2\omega^4}\ln\Big(1-\frac{r^2\omega^2}{N^2}\Big)\Big]+2 & \text{if} \quad r\le r_\omega \\ 0 & \text{if} \quad r>r_\omega . \end{cases} \tag{21}$$

The constants of integration have been fixed by imposing $f(0)=\pi$ and r_ω becomes the radius of the soliton and it is fixed as a function of ω, N and μ by the equation

$$-\frac{r_\omega^2\mu^2}{4\omega^2}+\frac{\mu^2}{4\omega^4}(r_\omega^2\omega^2-N^2)\ln\Big(1-\frac{r_\omega^2\omega^2}{N^2}\Big)+2=0 \tag{22}$$

We note that even if time dependence generates a kinetic term in the Lagrangian which breaks the invariance under area preserving diffeomorphisms, we still obtain analytical solutions to the problem (with the exception of finding numerically the value of r_ω for given values of μ, ω and N).

When ω goes to zero we regain the static solution (15). r_ω as a function of ω has to be obtained numerically but this does not present any problem since r_ω is a "well behaved" decreasing function of ω. Figure 1 shows the function $f(r)$ for several values of ω. As ω increases r_ω decreases moderately until ω reaches $\omega_{max}=\sqrt{N\mu/2\sqrt{2}}$ where $r_\omega=r_{max}=\sqrt{2\sqrt{2}N/\mu}$. After this point there does not exist a baby-Skyrmion type solution. $f'(r)$ is discontinuous at r_{max}.

We compute the classical energy E as a function of ω and compare it with the same quantity in the case of the rigid body. The dynamical moment of inertia is defined by

$$I(\omega)=2\pi\int_0^{+\infty}dr\ r\sin^2 f(r)f'(r)^2 \tag{23}$$

which is 2 T/ω^2 where T is the kinetic energy of the system (see the first term in (19)). Replacing for $\sin f(r)f'(r)$ with the solution (21) and making the change of variables

$$\begin{aligned} r&=\sqrt{\tfrac{N}{\mu}}\rho \\ \omega&=\sqrt{N\mu}w \end{aligned} \tag{24}$$

we obtain

$$I(\omega(w)) = \frac{\pi}{2}\int_0^{\rho_w} d\rho\ \rho^3 \left(\frac{\rho^2 - \rho_w^2}{1 - w^2\rho^2}\right) \tag{25}$$

where ρ_w is a solution of the equation (obtained from (22))

$$\rho_w^2 w^2 + (1 - \rho_w^2 w^2)\ln(1 - \rho_w^2 w^2) - 8w^4 = 0 \tag{26}$$

and which is independent of N and μ. ρ_w ranges from 2 to $\sqrt{2\sqrt{2}}$ while w ranges from 0 to $w_{max} = 1/\sqrt{2\sqrt{2}}$. The moment of inertia of the static baby-Skyrmion is defined as

$$I_0 = \lim_{\omega \to 0} I(\omega). \tag{27}$$

This amounts to replacing the static solution for $f(r)$ in (23) and the result $I_0 = 16\pi/3$. $I_{max} = I(\omega_{max}) = 8\pi$. The energy of a rigidly rotating baby-Skyrmion is given by

$$E_N(\omega) = \frac{4}{3}4\pi\mu|N| + \frac{8\pi}{3}\omega^2 \tag{28}$$

while the true rotating baby-Skrymion's energy is:

$$\begin{aligned} E_N(\omega) &= \tfrac{\pi\mu^2}{16\omega^2}\left[112N^2 - 7\mu^2 r_\omega^4 - 16 r_\omega^4\omega^2\right] \\ &= \tfrac{\pi\mu|N|}{16w^2}\left[112 - 7\rho_w^4 - 16\rho_w^2 w^2\right] \end{aligned} \tag{29}$$

which is linear in μ and $|N|$. Figure 2 compares the rotational energy of a rigid baby-Skyrmion and the energy of the deformable baby-Skyrmion, and we note that the true energy is larger than the rigid rotor approximation for all values of ω in accord with the behaviour found by Piette *et al*[1]. This behaviour is as expected since, for example, a block of rubber would react to a steady rotation by increasing its energy faster than a rigid rotor. The surprising behaviour that we find is that r_ω is a decreasing function of ω and there is a ω_{max} after which there is no solution of this type. Even though the size of the baby Skyrmion decreases, its energy and moment of inertia increase. Their densities tend to concentrate towards the outside (see Figure 3). We speculate that this is a precursor to breakup or the emission of some form of radiation. The energy density actually becomes discontinuous at r_{max} for $\omega = \omega_{max}$.

The general form of the energy of a spinning deformable body is written as

$$E(\omega) = \frac{1}{2} I(\omega)\, \omega^2 + U(\omega) + M \tag{30}$$

where $U(\omega) = V - M$, represents the potential energy stored in the particle as it reacts to centrifugal forces, $M = V|_{\omega \to 0}$ is the mass of the static baby-Skyrmion (16) and

$$V = \pi \int dr \left[\sin^2 f(r) f'(r)^2 \frac{N^2}{r} + 2\mu^2 r \Big(1 - \cos f(r) \Big) \right]. \tag{31}$$

In the case of the rotating physical baby-Skyrmion, this is a slowly increasing function of ω.

To compute the quantum rotational spectrum of the baby-Skyrmion we use the Bohr-Sommerfeld condition[10] on the action and energy of the system:

$$\begin{aligned} W(E_n) &= S[\tau(E_n)] + E_n \tau(E_n) \\ &= (2n+1)\pi\hbar \end{aligned} \tag{32}$$

where $\tau(E_n)$ is the period of the motion for a given energy E_n and S the corresponding action. Due to the stationary nature of the soliton, only the kinetic term contributes in (32), yielding the quantisation condition

$$\omega_n = \frac{1}{2I(\omega)} (2n+1)\hbar \tag{33}$$

ω_n is then obtained numerically as a root of equation (33) and then replaced in the expression of the energy. It is in the solution of equation (33) that the deformation of the soliton "interacts" with its rotation, representing the action of the system which chooses its shape to achieve a quantum state.

In the case of the rigid body, the energy rises as n^2. In taking account of deformations, $I(\omega)$ is larger than $I_0 = 16\pi/3$, making the ω_n smaller than the corresponding rigid body solution. Then the net effect of the deformation will be to shorten the gap between sequential energy levels. We show both spectra in Figure 4. We observe that the rigid body approximation is acurate for the first few levels, followed by a marked divergence. Since there does not exist baby-Skyrmion type solutions for $\omega \geq \omega_{max}$ the rotational energy spectrum

of Figure 4 contains a finite number of quantum levels for the soliton. The number of those levels depends on μ and N and is easily shown to be:

$$n_{max} = \text{Integer Part}\left[\frac{1}{2}\left(16\,\pi\sqrt{\frac{N\mu}{2\sqrt{2}}} - 1\right)\right). \tag{34}$$

In the case of the full baby-Skyrme model, Piette *et al.*[1] found the classical rotational spectrum of the baby-Skyrmion. They showed that two regimes existed according to the relative size of the angular velocity ω and pion mass μ: if the baby-Skyrmion spins too quickly the solution of the equation of $f(r)$ becomes oscillatory at infinity, rendering the baby-Skyrmion's energy and moment of inertia infinite. They also showed that a baby-Skyrmion "cranked" that high will quickly reduce its energy and angular velocity by emitting pion radiation until $\omega \lesssim \mu$, which is the "steady rotation" regime. In our case, the size of the baby-Skyrmion actually decreases with ω but the polynomial profile prevents any oscillatory or long range behaviour of the function $f(r)$. There is an ω_{max} after which there is no solution of this form.

We end this section by stressing that the quantum states so obtained are not eigenstates of spin $\vec{S}$, nor isospin $\vec{I}$, but of the sum $\vec{S}+\vec{I}$ of these operators. This is due to the fact that the baby-Skyrmion is only invariant under the diagonal group spin/isospin. Thus the tower of states constructed would be exactly the same if we had considered iso-rotations about the 3 axis. The quantization we have followed treats the solitons as bosons. A more general quantization as anyons is possible, as put forward by Wilczek and Zee[9] and requires the inclusion of a Hopf term. We will not elaborate on this possibility here.

4. Closed string in 2+1 dimensions

In a previous article[7] we introduced the ansatz for extended solutions resembling strings or strips, in the complete baby-Skyrme model and related models. Here we shall construct a physical state with such strings and compute its energy. The simplest "physical state" one can construct with the string is an annulus made by bending the string and closing it. We refer the reader to ref. [7] for the details and notations. To do this, one has to re-interpret

the coordinate along the string, y, as an angle which winds around the origin. The y axis is mapped to a circle of radius A and the x coordinate becomes radial. The ansatz that we use is

$$\vec{\phi}_< = (\sin f_<(r)\cos N\theta, -\sin f_<(r)\sin N\theta, \cos f_<(r)) \quad \text{for} \quad r < A \tag{35}$$

$$\vec{\phi}_> = (\sin f_>(r)\cos M\theta, \sin f_>(r)\sin M\theta, \cos f_>(r)) \quad \text{for} \quad r > A \tag{36}$$

where $\vec{\phi}_<$ represents the inner part of the annulus ($r < A$) and $\vec{\phi}_>$ the outer part ($r > A$). We have deliberately generalised the ansatz by winding $\vec{\phi}_<$ N times and $\vec{\phi}_>$ M times around the origin. There is no boundary problem since $f(r = A) = \pi$ no matter the values of N and M. The baryon number of the annulus is the sum $N + M$ where N and M are both non-zero integers (although their sum can be zero).

We now compute the energy and profile for the annulus. The procedure to follow is exactly the same as before with the exception that we have two functions to compute: one, $f_<$, for the inner part of the annulus, and the other, $f_>$, for the outer part. The corresponding differential equations are coupled together by the parameter A. One finds the "inner" solution $f_<$:

$$f_<(r) : \begin{cases} 1 - \cos f_<(r) = \frac{r^4\mu^2}{8N^2} + r^2\frac{\mu^2}{4N^2}\Big(\frac{4|N|}{\mu} - A^2\Big) + \frac{A^4\mu^2}{8N^2} - \frac{\mu A^2}{|N|} + 2 \text{ if } A \geq r \geq r_< \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad r < r_< \end{cases} \tag{37}$$

where $r_< \equiv \sqrt{A^2 - \frac{4|N|}{\mu}}$ and the "outer" solution $f_>$:

$$f_>(r) : \begin{cases} 1 - \cos f_>(r) = \frac{r^4\mu^2}{8M^2} - r^2\frac{\mu^2}{4M^2}\Big(\frac{4|M|}{\mu} + A^2\Big) + \frac{A^4\mu^2}{8M^2} + \frac{\mu A^2}{|M|} + 2 \text{ if } A \leq r \leq r_> \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad r > r_> \end{cases} \tag{38}$$

where $r_> \equiv \sqrt{A^2 + \frac{4|M|}{\mu}}$. This solution has energy

$$E_{N,M} = \frac{4}{3}4\pi\mu\Big(|N| + |M|\Big) \tag{39}$$

and area

$$\begin{aligned} A_{N,M} &= \pi\,(r_>)^2 - \pi\,(r_<)^2 \\ &= \tfrac{4\pi}{\mu}\Big(|N| + |M|\Big) \end{aligned} \tag{40}$$

which is independent of the parameter A, for the same reason that the area of the string was independent of its length L. Indeed a large radius A implies a very thin annulus while a smaller A implies a thicker annulus. Finally there is a minimum A for which $r_< = 0$ and the annulus becomes a disc however with a different configuration inside with respect to the baby-Skyrmion.

Before ending this section let us make a few remarks. First, the energy of the solution does not change if we "move" the baryon number from the inner part of the annulus to the outer as long as the total $N + M$ does not change (for both N and M positive). This generates another degeneracy in the energy spectrum.

Second, in order for $r_<$ to be real, one has to ensure that $A^2 \geq 4|N|/\mu$, implying that a solution of large baryon number must have a large radius A, or that most of its baryon number (and energy) must be stored in the outer shell of the annulus.

Third, the state with $M + N$ is different from the baby-Skyrmion with $B = N + M$ in that it actually contains the vacuum in its interior: $f_< = 0$ for $0 < r < r_<$. We see that in the $B = 2$ sector for example, we have 2 different lumps degenerate in energy: the $N = 2$ baby-Skyrmion, or "deuteron", and the $N = M = 1$ annulus. We remind the reader that the deuteron cannot be considered analogous to the annulus since $f = \pi$ at its center. We will give an indication in the next section as to the importance of these states in baby-Skyrmion scattering.

5. Considerations on soliton scattering

We use the method of Manton[11] while exploring the symmetries of the deuteron state in the usual 3+1 dimensional Skyrme model and the symmetries of scattering states. His method was based on the fact that looked from a distance, a Skyrmion resembles a triad of orthogonal dipoles. Piette *et al*[1,3] showed that similarly baby-Skyrmions look like a pair

of orthogonal dipoles. One way to see this is to use the $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$ parametrisation and the constraint $\vec{\phi}^2 = 1$. Using the constraint to eliminate, say, ϕ^3 one can describe a baby-Skyrmion configuration completely with the fields ϕ^1 and ϕ^2. Let us now consider the region of the plane defined by $x \sim 0, y > 0$. Since the baby-Skyrmion field is radial when projected in the $(1,2)$ plan, in this region ϕ^1 will be very small, while ϕ^2 will dominate and be positive. In the opposite region ($x \sim 0, y < 0$), it is the same situation but now ϕ^2 dominates while being negative. So the field ϕ^2, which falls at infinity as an exponential in the full model, represents from a far, a dipole oriented parallel to the y axis. The same is valid for ϕ^1 except along the x axis. The baby-Skyrmion is thus constructed from a pair of orthogonal dipoles (see[2,3] for a more rigorous demonstration of the dipole nature of the field of the baby-Skyrmion).

It was Manton's[11] idea to show that the natural evolution of these dipoles during a Skyrmion-Skyrmion scattering with relative orientation 180° (the so-called most attractive channel) is compatible with a 90° scattering. This was based on the observation that the discrete symmetries under reflection of the Skyrmion fields were conserved throughout the scattering. He also put forward the hypothesis that the deuteron state, which is the expected intermediate state of the scattering, possesses a cylindrical symmetry around the axis perpendicular to the scattering plane, englobing the discrete symmetries mentioned.

A similar reasoning should also be valid in the full baby-Skyrme model, and numerical simulations have concurred with the 90° scattering and an intermediate state made up of a $N = 2$ baby-Skyrmion for the case of 2 baby-Skyrmions with a relative iso-rotation of 180° around the z axis[1]. This is also completely consistent with Mackenzie's[12] theorem which proves under very general and reasonable assumptions that head on, geodetic scattering of identical particles must be through 90° if the coincident point is attained. Scattering with different relative orientations has also been studied, as we will discuss in more detail later.

In our case, we do not have any dipole picture since our solitons have finite size and the "long range" interactions are not well approximated by a linear theory of massive or

massless pions. What we have here are well determined regions where ϕ^1 or ϕ^2 dominate clearly (we represent these regions by a circle containing a "+" or "-" sign, as in Manton's notation. See Figure 5 for a picture of a baby-Skyrmion, and one that has been rotated by 180° around the z axis). This does not keep us from using Manton's framework. If anything, the finite and exactly free nature of our solitons render exact for a pair of baby-Skyrmions the analog of the approximate, discrete symmetries of a pair of usual Skyrmions, even when the baby-Skyrmions are at a finite distance from one another (in fact, anytime that they don't actually touch each other).

We now turn to the scattering of baby-Skyrmions without a relative iso-rotation. This case was not discussed by Manton, but treated numerically by Piette *et al.*[1] for the full baby-Skyrme model. They find 0° or 180° scattering. These cases are actually indistinguishable from just the scattering data! Figure 6 shows the "dipole image" of a pair of baby-Skyrmions without any relative iso-rotation for both fields ϕ^1 and ϕ^2. One can easily verify that this configuration transforms according to

$$\begin{aligned} x \to -x : \phi^1 &\to -\phi^1 \qquad\qquad & y \to -y : \phi^1 &\to \phi^1 \\ \phi^2 &\to \phi^2 & \phi^2 &\to -\phi^2 \end{aligned} \tag{41}$$

which are actually just the symmetries of a single baby-Skyrmion. These symmetries are not shared by a pair of baby-Skyrmions moving away along the y axis so we conclude that the most probable scattering process is 180°/0° collisions. But what could be the intermediate state of the collision? We know that it cannot be the deuteron since it does not transform according to (41). In fact the only $B = 2$ state we know of which possesses the right symmetries is the $B = 2$ annulus described in the preceding section, wich is degenerate in energy with both the deuteron and a pair of separated $B = 1$ baby-Skyrmions. We might then speculate that this annulus could be the intermediate state arising in the 180° or 0° scattering. It would be interesting to actually compute the energy of this annulus state in the full baby-Skyrmion model and to look for this state by studying numerically the scattering between unrotated baby-Skyrmions.

6. Baby-Skyrme type model on a curved space

We end this article by briefly studying the properties of the baby-Skyrmion of our model when put on a curved space, namely on a 2-sphere of radius R parametrized by the angles θ and ϕ where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Our aim is to study the properties of a dense system of baby-Skyrmions, like when put on a regular 2-dimensional array. The same procedure was followed by N.S. Manton *et al* [13] in the case of the usual Skyrme model in 3+1 dimensions without a mass term, where ordinary space was compactified into a 3-sphere. They showed that below a critical radius of the 3-sphere, the solution of lowest energy is not the Skyrmion but a configuration with uniform energy density, and recognized there chiral symmetry restoration. This makes sense since the Skyrme model is related to QCD in the limit of a large number of colours.

We will now compute the energy of the baby-Skyrmion for decreasing values of the 2-sphere radius, and compare it to the energy of the configuration with constant energy density.

When replacing the ansatz for the baby-Skyrmion on a 2-sphere

$$\vec{\Phi}(\theta, \phi) = \Big(\sin f(\theta) \cos\phi, \sin f(\theta) \sin\phi, \cos f(\theta)\Big) \tag{42}$$

into the equation of motion of the Lagrangian on a space-time of metric $g_{\mu\nu}$ (in our case $\sqrt{g} = R^2 \sin\theta$) given by

$$\partial_\mu \Bigg(\sqrt{g} \partial^\nu \vec{\Phi} \vec{\Phi} \cdot \Big(\partial^\mu \vec{\Phi} \times \partial_\nu \vec{\Phi} \Big) \Bigg) + \sqrt{g} \mu^2 \vec{\Phi} \times \hat{n} = 0 \tag{43}$$

we find the following analytical solution:

$$f(\theta) : \begin{cases} 1 - \cos f(\theta) = \frac{R^4 \mu^2}{2N^2} (\cos\theta_0 - \cos\theta)^2 & \text{if} \quad \theta \le \theta_0 \\ 0 & \text{if} \quad \theta > \theta_0 \end{cases} \tag{44}$$

where θ_0 is the "size" of the solution on the 2-sphere and is given by

$$\sin \frac{\theta_0}{2} = \frac{1}{R} \sqrt{\frac{|N|}{\mu}}. \tag{45}$$

This solution has again compact support and finite size and can only exist for $R\sqrt{\mu/|N|} \geq 1$ in order for θ_0 to be real. $R\sqrt{\mu/|N|} = 1$ represents the situation where the soliton entirely covers the 2-sphere and continues to satisfy $\sin f(\theta_0) f'(\theta_0)$ equal to 0 with $\theta_0 = \pi$. The energy of the solution is obtained by replacing equation (44) in the general expression for the energy in curved space. We find:

$$E = \frac{4}{3} 4\pi\mu|N| \tag{46}$$

which is surprisingly independent of R, and hence, equal to the energy of the baby-Skyrmion in flat space.

If we want to reduce still the radius of the 2-sphere, we have to relax the constraint that $\sin f(\theta) f'(\theta)$ be 0 at the edge of the soliton. The derivative will then be discontinuous at that point but the energy, being only function of the first derivative of $f(\theta)$, will not diverge. The so-called "compressed" baby-Skyrmion solution, existing only for $R\sqrt{\mu/|N|} \leq 1$, is also obtainable analytically:

$$1 - \cos f(\theta) = -\frac{R^4\mu^2}{2N^2} \sin^2\theta + 2\cos^2\frac{\theta}{2} \qquad \theta \in [0, \pi] \tag{47}$$

This solution completely fills the 2-sphere and has energy:

$$E = \frac{2\pi N^2}{R^2} + 4\pi\mu^2 R^2 - \frac{2\pi\mu^4 R^6}{3} \tag{48}$$

which diverges if $R \to 0$. The former happens because of the 4 derivatives of the Skyrme term combined with the integration measure and illustrates that a point soliton will have infinite energy. Of course, the energy of the two solutions will become equal in the limit where $R\sqrt{\mu/|N|} = 1$.

We now want to compare the energy of these solutions with the energy of the configuration with uniform energy density. Because of the presence of the mass term in the Lagrangian, this type of configuration is not a solution of the equations of motion. This was not the case in the work of Manton *et al*[13] since they did not include any mass term to stabilize the solitons against scale changes of the coordinates. In our case, with $f(\theta) = \pi - \theta$ and the baby-Skyrmion ansatz, we find the following energy:

$$E = \frac{2\pi N^2}{R^2} + 4\pi\mu^2 R^2 \tag{49}$$

which is always higher in energy than the baby-Skyrmion. We see that there is no phase transition between the two states as R changes values.

We thank R. Mackenzie and W.J. Zakrzewski for useful discussions, W.C. Chen for help with the numerical work, and the referee for his important criticisms. This work supported in part by NSERC of Canada and FCAR of Québec.

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FIGURES

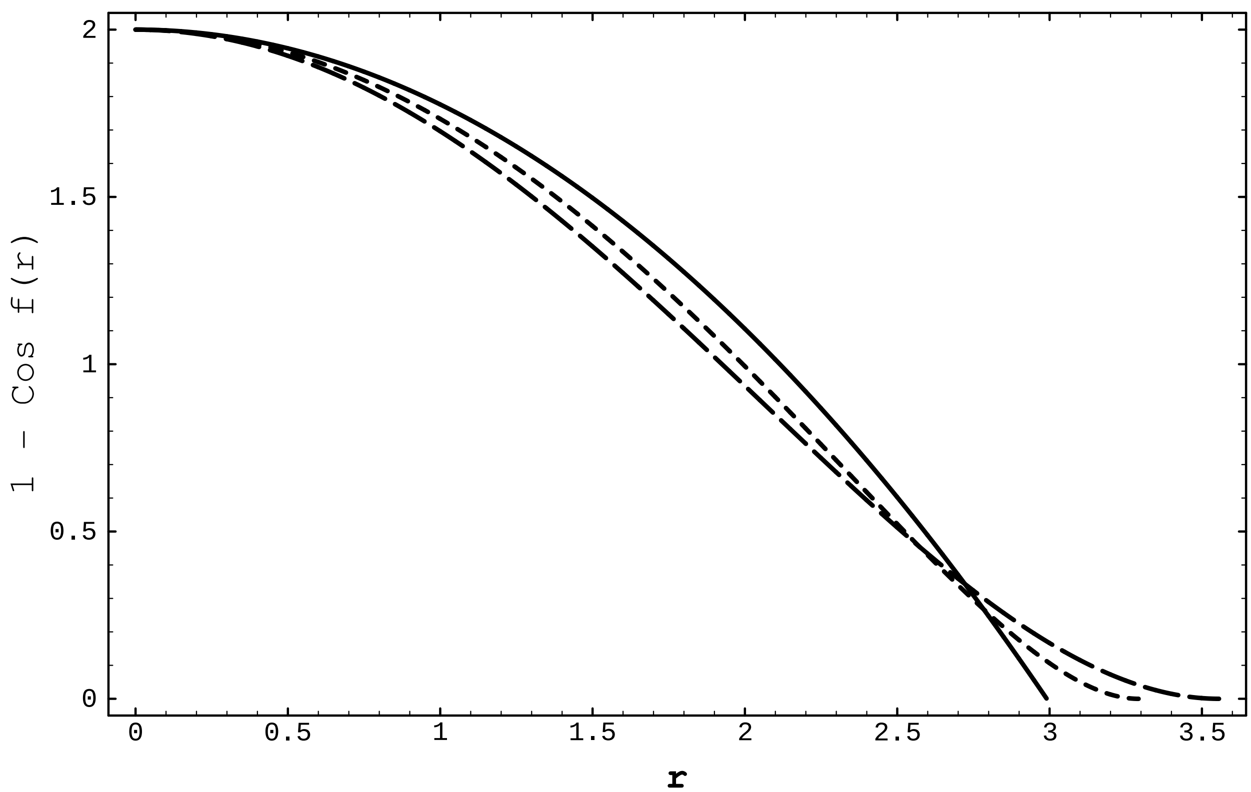

Fig. 1. Profile of the function $f(r)$ of a baby-Skyrmion spinning at the angular velocity 0 (long dashed line), 0.25 (dashed line) and ω_{max} (solid line), with $N = 1$ and $\mu^2 = 0.1$.

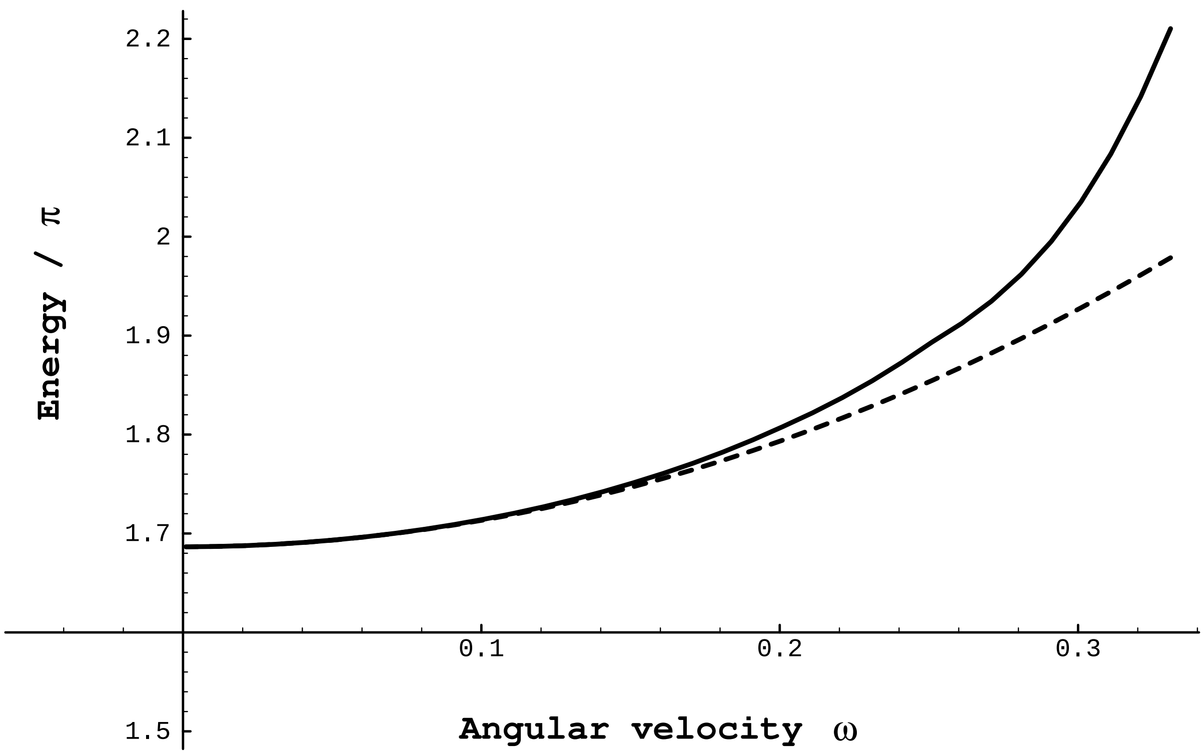

Fig. 2. Plots for $N = 1$ and $\mu^2 = 0.1$ of the energies of a baby-Skyrmion spinning at angular velocity ω ranging from 0.001 to ω_{max}, in the rigid body approximation (dashed line), and when centripetal deformations are taken into account (solid line).

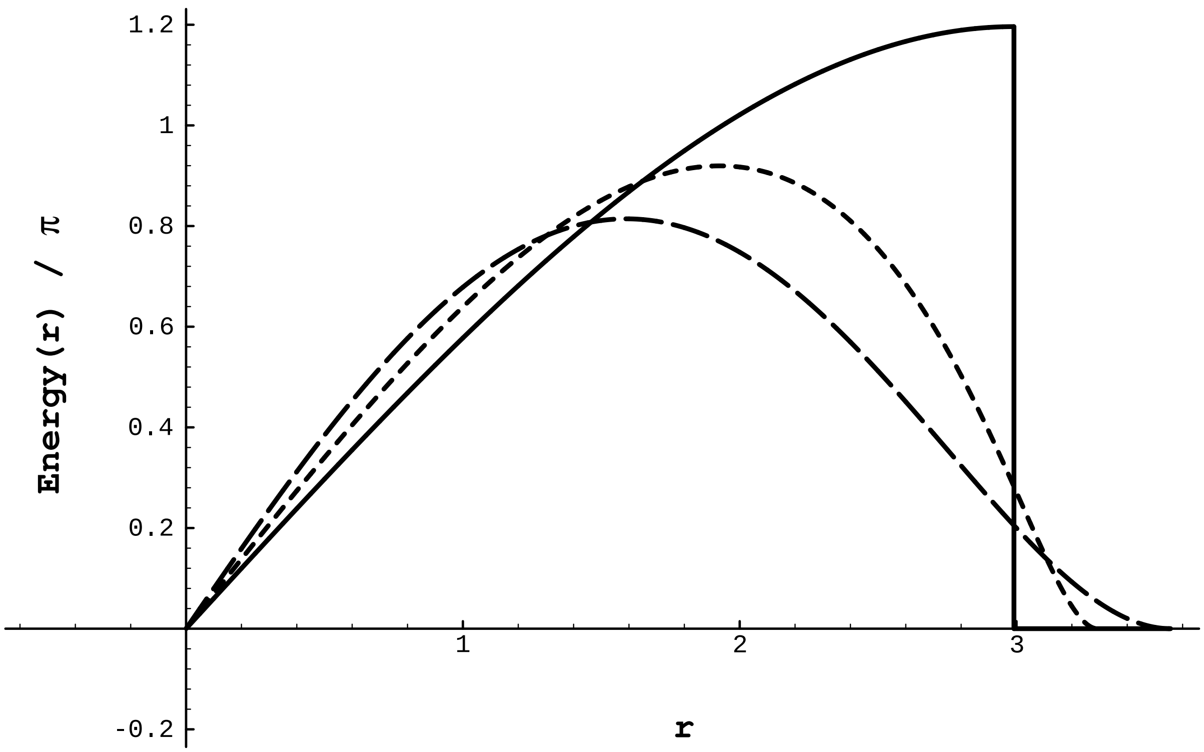

Fig. 3. Plot of the radial energy distribution of a baby-Skyrmion spinning at angular velocities 0 (long dashed line), 0.25 (dashed line) and ω_{max} (solid line), for $N = 1$ and $\mu^2 = 0.1$.

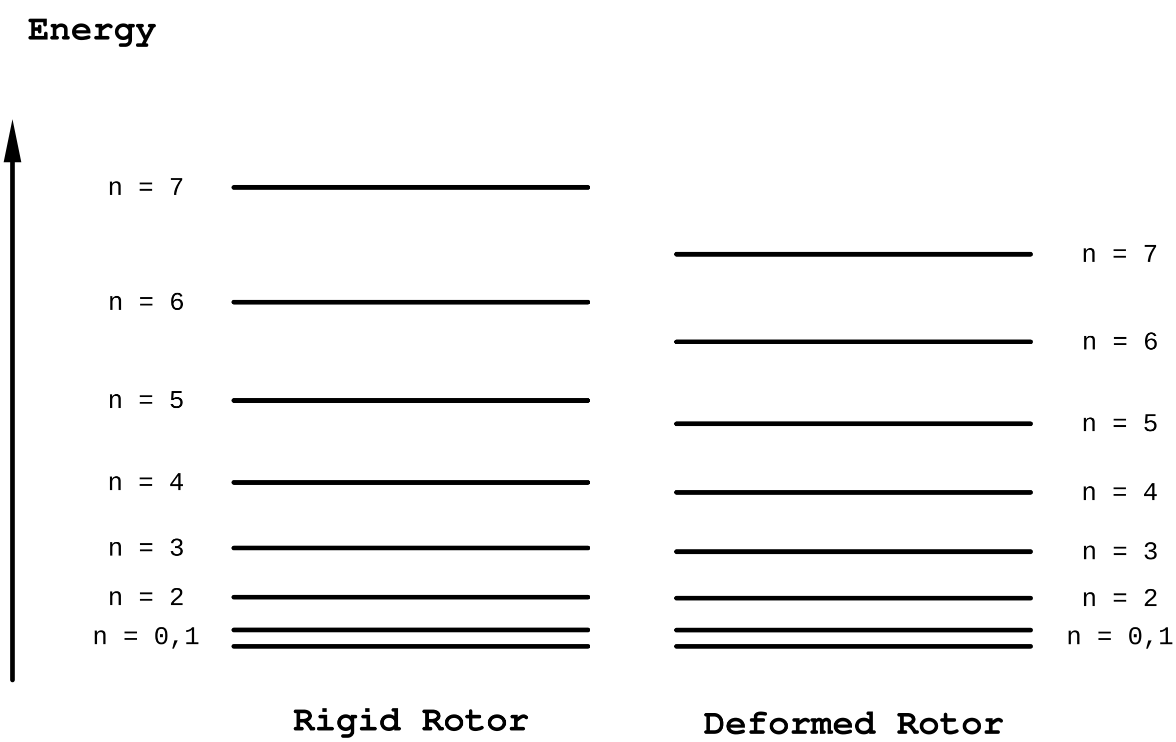

Fig. 4. Comparison of the quantum rotational energy spectra for a baby-Skyrmion with $N = 1$ and $\mu^2 = 0.1$, in the rigid rotor approximation and for the exact, deformed rotor.

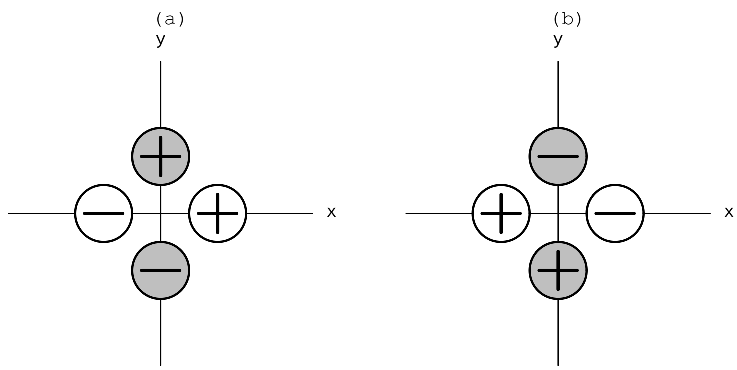

Fig. 5. Representation of the baby-Skyrmion field using Manton's notation of dipole regions. The regions where the field ϕ^1 dominates are drawn in white, while those where ϕ^2 dominates are drawn in gray. The figure (a) represents a baby-Skyrmion unrotated, while the figure (b) represents one that has been rotated by 180° around the z axis.

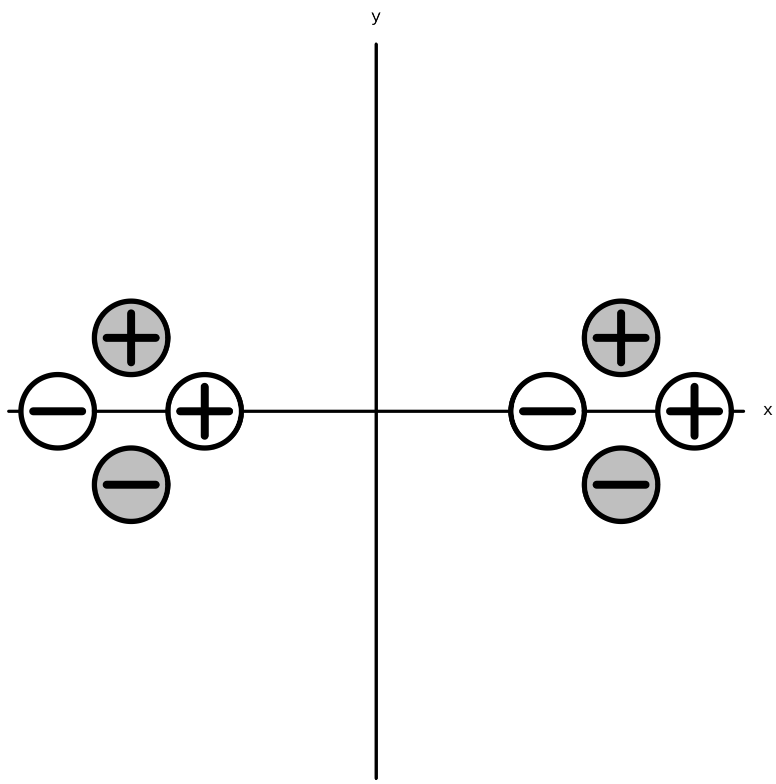

Fig. 6. Field of baby-Skyrmions with no relative iso-rotation.